1. (a) Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of one of the following types:

(i) $\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases}$ i.e. $g = \begin{array}{c} \quad\quad\quad j \\ \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & a & \\ & & & 1 \end{bmatrix} \end{array}$

(ii) $\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$ i.e. $g = \begin{array}{c} \quad\quad k \quad\quad j \\ \begin{bmatrix} 1 & & & & \\ & 1 & & 1 & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \end{array}$

(iii) $\begin{cases} g(e_k) = e_k & k \neq i, j \\ g(e_i) = e_j \\ g(e_j) = e_i \end{cases}$ i.e. $g = \begin{array}{c} \quad\quad i \quad\quad j \\ \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \end{array}$

If $U$ is a rectangle, show that the volume of $g(U)$ is $|\det g| \cdot vol(U)$.

(b) Prove that $|\det g| \cdot vol(U)$ is the volume of $g(U)$ for any linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$. (*Hint: If* $\det g \neq 0$, *then* $g$ *is the composition of linear transformations of the type considered in (a).*)

Sol: (a). (i). We may assume $U = [0,1]^n = e_1 \times e_2 \times \cdots \times e_n$

$vol(g(U)) = |e_1| \times \cdots \times |ae_j| \times \cdots |e_n| = |a| = |\det g| \cdot vol(U)$

(ii) Note $g(U) = e_1 \times \cdots \times \hat{e}_j \times \cdots \times \hat{e}_k \times \cdots \times e_n \times g(e_j \times e_k)$

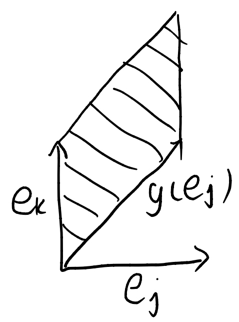
By Fubini Thm,

$$vol(g(U)) = \int_{e_1 \times \cdots \times \hat{e}_j \times \cdots \times \hat{e}_k \times \cdots \times e_n} \left( \int_{g(e_j \times e_k)} dx_j \, dx_k \right) dx_1 \cdots \widehat{dx_j} \cdots \widehat{dx_k} \cdots dx_n$$

$$= 1 \cdot \left( \int_{g(e_j \times e_k)} dx_j \, dx_k \right)$$

$$= 1$$

$$= |\det(g)| \cdot vol(U)$$

$g(e_j \times e_k) =$ 

(iii) $g(U)$ is still a rect with edges $e_1 \cdots e_n$ but of different order, so $vol(g(U)) = 1 = |\det(g)| vol(U)$

(b) Suppose $g = g_i \circ g_{ii} \circ g_{iii}$. Then

$$vol(g(U)) = vol(g_i(g_{ii} \circ g_{iii}(U)))$$

$$\overset{(a)}{=} |\det g_i| \, vol(g_{ii} \circ g_{iii}(U))$$

Similarly,:

$$= |\det g_i| \cdot |\det g_{ii}| \cdot |\det g_{iii}| \, vol(U)$$

$$= |\det \underbrace{g_i \circ g_{ii} \circ g_{iii}}_{g}| \, vol(U)$$

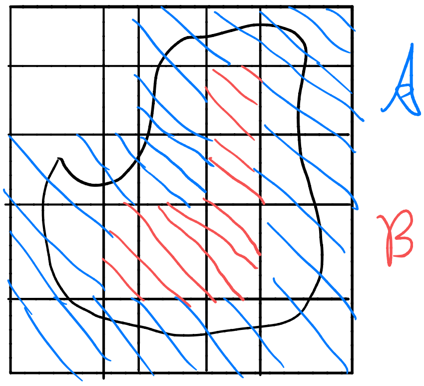
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with measure zero $\partial\Omega$. Show that for any $\epsilon > 0$, there exists a compact subset $K \subset \Omega$ such that $\partial K$ has measure zero and $\text{Vol}(\Omega \setminus K) < \epsilon$.

Recall (Integrals on arbitrary regions):

Let $\chi_\Omega(x) := \begin{cases} 0, & x \notin \Omega \\ 1, & x \in \Omega \end{cases}$. $R \supset \Omega$ be a rect. Then $\int_\Omega f := \int_R f \cdot \chi_\Omega$

$\chi_\Omega$ is integrable iff $\partial\Omega$ has measure 0

(Spivak, P55)




In this case, $vol(\Omega) = \int_\Omega 1 = \int_R \chi_\Omega$.

Sol: Given a partition of $R$, let $A := \{\text{rect intersecting } \partial\Omega\}$.

Remark: A partition is always finite.

$B := \{\text{rect} \subset \Omega \text{ not intersecting } \partial\Omega\}$

$\left.\begin{array}{l}\Omega \text{ is bounded} \Rightarrow \partial\Omega \text{ is compact} \\ \partial\Omega \text{ has measure } 0\end{array}\right\} \Rightarrow \partial\Omega \text{ has content } 0$

$\Rightarrow \exists$ a partition $P$ s.t. $vol(\bigcup A) < \varepsilon$

Let $K := \bigcup \mathcal{B}$. Note $\chi_{\Omega \setminus K} \equiv 0$ on $K$.

$$
\begin{aligned}
\operatorname{vol}(\Omega \setminus K) &= \int_R \chi_{\Omega \setminus K} \\
&\leq U(\chi_{\Omega \setminus K}, P) \\
&= \sum_{A \subset \mathcal{A}} \sup_A (\chi_{\Omega \setminus K}) \cdot \operatorname{vol}(A) \\
&\leq \sum_{A \subset \mathcal{A}} \operatorname{vol}(A) \\
&< \varepsilon
\end{aligned}
$$

Finally, $\Omega$ is bounded $\Longrightarrow$ $K$ is a finite union of rect hence compact.